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Initial Understandings of Fraction Concepts Evidenced by Students With Mathematics Learning Disabilities and Difficulties: A Framework

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Abstract

Documenting how students with learning disabilities (LD) initially conceive of fractional quantities, and how their understandings may align with or differ from students with mathematics difficulties, is necessary to guide development of assessments and interventions that attach to unique ways of thinking or inherent difficulties these students may face understanding fraction concepts. One way to characterize such conceptions is through the creation of a framework that depicts key understandings evidenced as students work with problematic situations. The present study extends current literature by presenting key understandings of fractions, documented through problem-solving activity, language, representations, and operations, evidenced by students with LD and mathematics difficulties as they engaged with equal sharing problems. Clinical interviews were conducted with 43 students across the second, third, fourth, and fifth grades. Results of the study suggest that students with LD hold similar informal notions of key understandings of fractions as students with mathematics difficulties and that many of the students evidenced rudimentary understandings of fractional quantities. Researchers discuss implications of the findings in relation to considerations for designing interventions to support and extend students' initial conceptions of fractional quantity.

Keywords

learning disability, framework, fractions, assessment, prior knowledge

The majority of 21st-century occupations require a college degree. Accordingly, the primary focus of mathematics education reform over the past 60 years has been on the successful completion of algebra (National Mathematics Advisory Panel [NMAP], 2008), a "gatekeeper" course (Bailey, Hoard, Nugent, & Geary, 2012) granting access to college and career readiness. Yet, research suggests that difficulties students have learning algebra may be explained by their limited understanding of more foundational concepts and skills. One of the most powerful links is found between algebraic concepts and an understanding of rational number—most markedly, fractions (Booth & Newton, 2012; NMAP, 2008; Siegler et al., 2012). An incomplete conceptual understanding of fractional quantities, in particular, could have an amassed effect on students' ability to operate with or apply computational procedures in fractional quantities in higher level mathematics contexts (Hackenberg, 2013; NMAP, 2008).

Despite its importance, a strong conception of fractions is notoriously difficult for students to construct. Research suggests elementary school students labeled as having learning disabilities (LD), in particular, begin their study of fractions with diminished understandings compared with what is documented among their peers without disabilities (Hecht, Vagi, & Torgesen, 2007). Other research (e.g., Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013) suggests further distinctions, or gaps, in fractional knowledge between students with LD and students who experience difficulties learning mathematics but do not have disabilities. Yet, there is a dearth of information in the literature that explains the nature of the initial fractional knowledge elementary students with LD *do* hold, in what ways their fractional knowledge is diminished, or how students' initial understandings may differ from those of peers who struggle but do not have disabilities. Such information could be used to develop a framework that may prove useful for researchers and practitioners

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wishing to design instruction from students' unique, initial conceptions.

In the paragraphs that follow, we first review existing information regarding sources or causes of difficulties in initial fraction conceptions for elementary students with LD and how initial conceptions may be unique from students who struggle to learn but do not have disabilities. Next, we ground the documentation of students' initial conceptions as through their own activity while immersed in problematic situations and introduce equal sharing as a situation that can be used to both elicit and, later, extend initial conceptions. Then, we present a summary of literature outlining prior research of conceptions evidenced by students immersed in these situations as a means of promoting a guide for analysis of the thinking uncovered in the present study. Finally, research questions are presented.

Previous Explanations of Diminished Fractional Understandings: Students With LD

Many researchers assert that informal understandings elementary students with LD hold of fractions may be unique from those of elementary students who struggle but do not have disabilities. Yet, competing explanations are named. For instance, some research suggests that the uniqueness may rest in broad cognitive deficits intrinsic to LD. These lines of research (e.g., Davis et al., 2009) attempt to link the difficulties students with LD have in learning fractions with limitations in inborn cognitive factors, such as working memory or processing. Yet, the effects of cognitive factors on conceptual understanding of fractions, specifically, were unclear, and the informal conceptions students with LD did possess were largely undefined in these studies. Nonetheless, researchers propose that cognitive deficiencies inherent to students with LD have an adverse effect on their ability to develop understandings, possibly resulting in reduced or developmentally unexpected notions of fractions.

Other researchers assert that the uniqueness rests in different interpretations or understandings of mathematical representations due to the LD (Lewis, 2010, 2014). For example, understanding shaded parts within continuous area representations of fractions as "taken away" from the whole and understanding a partitioned part, such as ½, as the action of partitioning (as opposed a quantity) were described as unique understandings evidenced by adults labeled as LD. This research delineated conceptions that students with LD seemed to possess, yet it is difficult to frame the conceptions as initial because they were documented in adult learners. That is, it is unclear whether elementary children with LD would display similar conceptions in their initial knowledge. It is also unclear whether such conceptions were unique to disability or linked to prior instructional experiences (or the lack thereof). Furthermore, researchers in mathematics education document similar conceptions among students labeled as low achieving or typically achieving but not LD. These researchers name underdeveloped mathematics, such as multiplicative unit coordination or abstracted notions of number composite structures, as possible reasons for not seeing a fraction apart from yet related to a whole (as opposed a function of a disability; Hackenberg, 2013; Olive & Vomvordi, 2006; Tzur & Lambert, 2011).

Documentation of Students' Initial Conceptions Through Activity

The ambiguity of the unique initial knowledge elementary students with LD hold of fractions, and if or how their conceptions set them apart from those of students who struggle but do not have disabilities, warrants future research. Such documentation seems a necessary stepping-stone to guide development of assessment and intervention from how these children might begin to conceive of fractions (Vukovic, 2012). However, competing explanations of the origins of difficulties students have learning fractions and basing the explanation on how students are thought to be deficient makes for a questionable start. Another way to provide the documentation is through the creation of a framework that depicts students as capable and maps how they *do* initially conceive of fractions related to key understandings that support later development of fractions as quantities. The documentation begins with research that utilizes students' activity within problematic situations as a platform to document their initial conceptions (Tzur, Johnson, McClintock, & Risley, 2012).

Specifically, student's activity within equal sharing tasks—equally sharing an object or objects among different numbers of people, where the result is a fractional quantity—can evoke a variety of problem-solving activity, language, representations, and observable operations reflective of initial notions that build toward key understandings of fractions as quantities (Piaget, Inhelder, & Szeminska, 1960). We choose to utilize equal sharing situations to document student's conceptions for three reasons. First, the part–whole interpretation (e.g., partitioning a given whole into four pieces, shading one, and naming the fraction as one fourth), while traditional and widely used in U.S. curriculums, does little to help students recognize fractions as specific, unique quantities (Booth & Newton, 2012; Tzur, 1999). Second, many children have not had instructional experiences specific to equal sharing in school, so these experiences are likely novel to children and thus an appropriate context to gauge initial or informal knowledge. Finally, equal sharing situations seem to trigger students' informal ways of reasoning that can be a basis for developing increasingly sophisticated fractional knowledge (Empson, Junk, Dominguez, & Turner, 2006; Steffe & Olive, 2010; Tzur, 2007).

Although there are many ways to define and document initial notions of fractions through equal sharing (Charles & Nason, 2000; Empson et al., 2006; Kieran, 1976; Pothier & Sawada, 1983; Steffe & Olive, 2010; Streefland, 1993; Tzur, 1999, 2007), we utilized existing research from Tzur (1999, 2007), Wilkins, Norton, and Boyce (2013), Empson and colleagues (Empson et al., 2006), and Streefland (1993) to design equal sharing situations to uncover students' understandings in this study. The approaches used by these researchers have been found as promising for not only uncovering informal knowledge but also developing strong, quantitative notions of fractions as quantities for all students (Lamon, 2007), including students with LD. We synthesize the empirical research in the paragraphs below.

Initial Conceptions Evidenced in Equal Sharing Situations: Prior Research

Many researchers begin students' experiences in equal sharing in situations that promote consideration of the size of a share when one object, such as a French fry represented by a long strip of paper, is shared between varying numbers of people (Tzur, 1999, 2007). This type of situation requires students to estimate the size of the share for n sharers by using a single, disembedded item, such as a drinking straw, and repeating, or iterating, the item n times against the whole to verify that it is the correct size. Students use the operations of partitioning and iterating to ensure that the whole is exhausted (Tzur, 1999, 2007). Should the whole not be exhausted, or too few/many parts are created, students adjust the size of their original guess, and repeat the partitioning through iteration (Tzur, 1999).

Iterating eventually aids students to equally partition one whole by coordinating the iterated part with an exhaustion of a unitized whole (Tzur, 2007). Such activity also helps students not only conceive of a whole as "one" (n/n) but also a multiple of the unit fraction ($n \times 1 / n$), consisting of a certain number of copies of a same-sized unit, a two-level unit coordination (Hackenberg, 2013). The relative size and equality of the parts along with the relationship between the size of the parts and the number of parts relative to one whole are two key understandings named by Piaget and colleagues (1960) for conceiving of fractions as quantities. Subsequent situations (Wilkins et al., 2013) can then be used that supply students with an already created part with a request to reconstruct the whole as a further test of the multiplicative coordination.

Other researchers begin students' experiences in equal sharing in situations that promote consideration of a partitioning and distributing plan for m items shared among n people (e.g., sharing five cookies among three friends, Empson et al., 2006; Streefland, 1993). This type of situation requires students to utilize knowledge of number composites and multiplication to plan a partitioning of the items such that each person receives an equal share and all of the items are used. For example, in the aforementioned example, students may try to create unequal shares should they not yet conceive of partitioning a whole (Piaget et al., 1960), or they may give one whole cookie to each friend and partition the remaining parts into halves and then the remaining half into three parts. Yet, when asked to quantify each person's share, the student may say "three" or "two and one-half," because the size of the parts was not anticipated or coordinated in relationship to one whole, and may not in the student's mind be differentiated from the whole (e.g., the parts being different sizes does not matter; Piaget et al., 1960). Conversely, students may distribute a whole cookie to each student and partition the remaining two cookies each into three parts and distribute a part from each cookie to each person. These students use an a priori relation between the items and the number of sharers and developing multiplicative notions to anticipate a partitioning plan (Empson et al., 2006). These students see the partitioned item as coordinated with one whole and may quantify one person's share as "1 + 1/3 + 1/3."

Partitioning and distributing eventually aids students to synthesize understanding of multiplication, division, and ratio via measurement (Empson et al., 2006). Activities of this type help children to begin to coordinate multiplication and division in a way that emphasizes mathematical relationships, eventually coming to anticipate that the solution to any sharing situation is the fraction represented by the number of items in the numerator and the number of people in the denominator. Moreover, conceiving of and developing a plan for partitioning and exhausting the whole are two key understandings named by Piaget and colleagues (1960) for conceiving of fractions as quantities.

Research Questions

A delineation of initial conceptions of fractions that students with LD hold as they relate to key understandings that support quantitative notions of fractional quantities are largely absent in the literature along with how their conceptions may be unique from those of their peers who have difficulties learning mathematics. The current study extends current literature by presenting the informal understandings of fractions evidenced by 43 children with LD or difficulties learning mathematics in semi-structured clinical interviews. The research questions were as follows:

Research Question 1: What informal/initial understandings of fractions do students with LD and difficulties evidence through their employed problem-solving strategies, language, representations, and observable operations as they engage in equal sharing tasks?

Research Question 2: Do employed strategies, language, representation, and observable operations vary between children with LD and difficulties?

Table 1. Characteristics of Students.

Characteristic	LD (%) $N = 21$	Tier 2 (%) $N = 23$
Grade		
2	5%	13%
3	24%	13%
4	48%	31%
5	23%	43%
Gender		
Male	76%	57%
Female	24%	43%
Ethnicity		
Caucasian	10%	9%
Black	14%	30%
Hispanic	71%	61%
Burmese	5%	0%
Disability[a,b]		
LD, working memory	29%	0%
LD, processing	10%	0%
LD, LTM	5%	0%
LD, fluid reasoning	5%	0%
LD, comorbid	51%	0%

Note. LD = learning disabilities; LTM = long-term memory.
[a]Predominant cognitive difficulty at or below 15th percentile. [b]*Woodcock Johnson Test of Cognitive Abilities*, 2007 or Bateria III, 2007.

Method

Participants

Participants in the study included elementary students in the second, third, fourth, and fifth grades. For students with LD, we defined inclusion criteria for study participants as having a cognitively defined label of LD and individualized education program (IEP) goals in mathematics. For students who struggle with mathematics, we defined inclusion criteria as inclusion in "Tier 2" intervention programs (i.e., documented, pervasive performance issues in mathematics, yet the difficulties experienced are not cognitive in origin). Out of a possible 70 students who met inclusion criteria, 44 assenting students participated in the study whose parents provided informed consent. Twenty-one of the students had documented, cognitively defined LD (per school documentation) along with IEP goals in mathematics and 23 of the students were included in "Tier 2" intervention programs. Characteristics of the students are listed in Table 1.

Setting

All interviews took place in a small classroom in one elementary school located in a large urban city in the southern United States. Interview sessions generally lasted one and one-half hours; researchers used additional sessions as needed to complete problem tasks.

Problem Tasks

Researchers designed a set of seven problem situations for use in the study based on the aforementioned synopsis of prior research. Problem situations were based in the context of equal sharing; the number of sharers ranged from two to four and the number of objects shared ranged from one to nine. Table 2 lists the tasks.

Three situations were designed to elicit unit fractional values less than one (e.g., one third) through the context of sharing a "French fry" (Tzur, 2007). These tasks included sharing one whole item between two and three people. Another task asked students to rebuild the whole when given a share, or part (e.g., one fifth; Wilkins et al., 2013). These situations were used to elicit and assess each student's propensity to utilize partitioning and iterating operations to conceive of the whole as so many same-sized copies of a fractional unit and coordinate unit fractions with respect to one whole. Throughout all tasks, students' representational levels (i.e., tangible, figurative, or symbolic) and naming of the created quantity (in answer to "how much?" or "what do we call that?") were recorded.

Another four situations were designed elicit fractional values greater than one (i.e., number of items > number of sharers) and less than one (i.e., number of items < number of sharers) in a story context (e.g., Four friends share three ice cream bars. How much of an ice cream bar did each friend receive?). These tasks included sharing five wholes between two people, nine wholes between four people, three wholes between four people, and operating with an established fractional quantity to reason about a total (i.e., $2/3 \times \blacksquare = 6$. These situations were used to elicit and assess students' propensity to use a mentally planned partitioning of each whole item coordinated with the number of sharers, combine the created unit fractions, and quantify it as an equal share with respect to one whole, that is, three items shared by four people as $(1 \div 4) + (1 \div 4) + (1 \div 4)$. Throughout the tasks, students' representational level and naming of the created quantity in answer to "how much of a whole does each person get?" were recorded.

Study Design and Procedures

Data collection entailed semi-structured clinical interviews (Ginsburg, 1997) done with each student individually. Interviews took place in a small classroom equipped with large tables, manipulative materials (i.e., unifix cubes, paper rectangles), writing instruments, and paper. All interviews were audio and video recorded. The student and the interviewer read each problem orally. Suggestions or procedures for solving the problems were not presented. Instead, students were encouraged to solve each problem in a way that made sense to them—they could use the manipulative materials, paper and pencil, or no materials to aid them in reaching

Table 2. Six Interview Tasks.

Problem tasks
1. Please share this fry equally between the two of us. What do you call each part of the French fry? How do you know?
2. Suppose we want to share the fry among three people now. This time, you cannot fold or use a ruler. Show me the size of the share. How do you know it is the correct size? What do you call each part of the French fry? How do you know?
3. Here is one whole French fry. Here is one person's share. What part of the whole is the share? How do you know?
4. Harry and Larry order five sandwiches to share equally between them. How many sandwiches does each of them receive?
5. Four children share three large candy bars. Each child eats the same amount and they finish all three candy bars. How much of a candy bar does each child eat?
6. Four children share nine sticks of clay for a project. How many sticks of clay does each child use?
7. Each student uses 2/3 of a sandwich bun for his or her lunch. How many students were there if six sandwich buns were used?

a solution. The interviewer pressed students to explain and justify each of their solutions in an attempt to understand their thinking processes. The interviewer repeated answers/statements back to students to encourage elaboration. When students produced a representation, the researcher asked what the drawing or symbols represented. The researcher also took anecdotal notes during each interview conducted.

In general, task administration followed the order of the tasks presented in Table 2 although the order was altered when possible to guard against a testing effect and to produce a dynamic assessment of current conceptions. That is, interviews were designed to reveal as much as possible about each student's understanding and thus were dynamically adapted depending on responses. The interviewer also individualized the context of each problem situation to student preference. Each student was asked to solve all problems so that trends in thinking could be observed.

Data Analysis Procedures

Data analysis was done on four levels. The first level of analysis addressed the first research question and employed a constant comparison method to delineate elements of students' thinking/conceptions of fractions (Leech & Onwuegbuzie, 2007; Strauss & Corbin, 1997). The principal investigator (PI) and two graduate students reviewed the first four videotaped interviews as a team. The team inspected the tasks completed in each interview one at a time. For each task, researchers examined (a) the way in which students solved the problem (nuances within evidenced strategies along with the nature of associated conceptions led us to ultimately refer to ways of solving problems by levels), (b) observable operations employed (i.e., if/how the student partitioned the whole; if the student iterated a created part against the whole), (c) representations used, and (d) language used to quantify a created share. We then gave each element of students' thinking an initial code. Researchers also informally noted possible key understandings related to fractions (Piaget et al., 1960) that began to emerge in the data (discussed further below). As more tasks and interviews were coded, we carefully compared each new chunk of data (i.e., each problem solution) with data coded previously and searched for confirming and disconfirming evidence to ensure consistency and validity (Creswell, 2012; Leech & Onwuegbuzie, 2007). This led to the creation of an initial codebook.

Next, each researcher independently coded six more student's videotaped interview sessions using the initial codebook. Codes were then compared using peer debriefing and collaborative work (Grbich, 2012). The comparison resulted in a slight refinement of the codes and the corresponding codebook. The iterative process of coding, comparing, and refining continued through three additional rounds of independent coding (Leech & Onwuegbuzie, 2007) until all tasks in all interviews were coded.

Inter-rater reliability (IRR) was established for each element of student's thinking through an examination of 30% of all tasks coded across the clinical interviews (i.e., a random pull of 12 coded interviews). We determined IRR using Cohen's (1960) kappa. The statistic produces a possible range of agreement between −1 and +1 and is a strong gauge of observed agreement between coders as it corrects for agreement that would be expected by chance. The analysis (i.e., agreements, adjusted for chance divided by agreements + disagreements, adjusted for chance) yielded a kappa of 0.73 for problem-solving method and 0.73 for operations, suggesting substantial agreement among coders (Hallgren, 2012). The analysis yielded a kappa of 0.67 for representations and 0.58 for naming, suggesting adequate agreement among coders (Hallgren, 2012).

Classical content analysis was used to determine each student's dominant problem-solving method, operations, representational level, and naming/quantification across the interview tasks. This descriptive information about the data was complementary to the constant comparative analysis used earlier (Leech & Onwuegbuzie, 2007). For each student, researchers considered the mode for each indicator across the tasks as evidence of dominance. Researchers also utilized dominant indicators for each indicator to address the second research question.

The second level of analysis also addressed the first research question and used emergent coding (Grbich, 2012; Miles & Huberman, 1994) to document key understandings for each interview that emerged *across the tasks* during each student's interview during the constant comparative analysis for problem-solving strategy, language, representations, and operations. Researchers returned to the first 10 interviews coded and identified major categories of key understandings evidenced across the tasks within each interview. Identified key understandings aligned with Piaget and colleagues' (1960) account of children's fraction understandings and included (a) the child's notions of the whole as being divisible, (b) the child's partitioning plan, (c) the relation between partitions and parts created, (d) the child's exhaustion of the whole, and (e) the child's propensity to create equal parts. Key understandings were considered in terms of development (i.e., early, developing, or solidified) and were placed in a framework. Researchers then used the framework to independently code all remaining interviews to establish each student's overall level according to the framework. IRR was established for overall trajectory level for all interviews (n = 44) using Cohen's (1960) kappa. Ninety-two percent IRR was achieved. Numbers of students falling at each level were also quantified.

The third level of analysis addressed the second research question and employed Mann–Whitney U tests (Lomax & Hahs-Vaughn, 2013) to evaluate differences in the dominant problem-solving strategy, operations, representations, and language used by students with LD and students in Tier 2 as they solved equal sharing problems. The independent variable was LD status (LD vs. Tier 2); the dependent variables were problem-solving level, partitioning operations level, iteration operations (absent/present), employed representational level, and language level. Separate tests on the dependent variables were run for each grade level (Lomax & Hahs-Vaughn, 2013). The fourth and final level of data analysis also addressed the first research question and involved data visualization techniques (Ward, Grinstein, & Keim, 2010) to visually examine trends in reference to the key understandings across all students interviewed. Researchers prepared a heat map of all five coded key understandings (lowest level—orange, highest level—yellow) for each student/interview. The map was analyzed to examine which key understandings led development at various levels.

Results

Our aim in this study was to delineate the initial understandings of fractions held by students with LD and students with mathematics difficulties and to demarcate differences in initial conceptions held by these students. The first research question addressed the initial understandings of fractions students with LD and difficulties evidence through their employed problem solving, language, representations, and observable operations as they engage in equal sharing tasks along with any key understandings that seemed to emerge in their work. The second research question asked whether differences in employed strategy, language, representation, and observable operations exist between children with LD and difficulties.

Analyzed data are presented in two parts. First, results of the constant comparison analysis are presented in terms of level for problem solving, language, operations, representations found in the analysis. Results of nonparametric tests are described after each variable in terms of tests for differences between children with and without LD. Second, results of emergent coding in terms of the key understandings are defined and quantified by level, a framework of key understandings is delineated, and results of trend analysis are illuminated using a heat map.

Constant Comparison Analysis

Problem-solving levels. Qualitative analysis resulted in four levels of problem-solving activity: (a) *No Fractions*, (b) *Emergent Sharing*, (c) *Half*, and (d) *Emergent Relations/Coordination*. Examples and descriptions follow.

No fractions. In Level 0, or *No Fractions*, students did not create fractions. In the first three tasks (hereafter referred to as "Fry tasks"), students did not engage with sharing the item and tried to add more items to create whole number shares. In the last four tasks (hereafter referred to as story problems), students created unequal shares (e.g., five items shared by two people would result in one person receiving two and one person receiving three) or added/took away items to create whole number shares. Five percent of students with LD and 0% of students in Tier 2 evidenced *No Fractions* as their dominant problem-solving level.

Emergent sharing. In Level 1, or *Emergent Sharing*, students utilized guess and check or a whole number based "build up" to share. For instance, in the Fry tasks, students would count "one, two, three," from the left, making unequal parts. Students would partition but not iterate a part and did not exhaust the whole although students sometimes extended their counting across the whole until they created a number of unequal pieces. In the story problems, students dealt out whole number objects, counting by ones, until they encountered a leftover. At that point, students either continued their whole number count onto the object or used a rudimentary partitioning to share the item. Fifty percent of students with LD and 30% of students in Tier 2 evidenced *Emergent Sharing* as their dominant problem-solving level.

Half. In Level 2, or *Half*, students began to coordinate making equal shares with exhausting the whole "after the fact." Students' in activity plan usually became apparent in subsequent attempts to coordinate making same-sized parts and use up the whole (i.e., Fry tasks) or when dealing with "leftovers" (i.e., story problems). For example, in the Fry tasks, students would work from the middle or from the end points, trying to keep equal parts. They eventually exhausted the whole; visually adjusting the part until they were satisfied the parts were equal and exhausted the whole. In the story problem, students distributed halves until they encountered a leftover(s). At that point, students repetitively halved the leftovers, and finally partitioned the last piece by the number of sharers. Thirty percent of students with LD and 35% of students in Tier 2 evidenced *Half* as their dominant problem-solving level.

Emergent relation or coordination. In Level 3, or *Emergent Relation or Coordination*, students coordinated making equal shares with exhausting the whole and used it as a plan coming into activity. Put differently, students linked partitioning to the number of sharers. In the Fry tasks, students used an object to stand in for a part and tested it against the length of the whole. Students would eventually exhaust the whole, usually by adjusting the created part in some manner across the whole until it was the correct length. In the story problems, students either (a) partitioned each item by the number of sharers or (b) created a number of parts equal to the number of sharers, sometimes utilizing multiplication facts. Fifteen percent of students with LD and 35% of students in Tier 2 evidenced *Emergent Relation or Coordination* as their dominant problem-solving level.

Problem solving: Differences between groups. A Mann–Whitney *U* test was conducted to evaluate the null hypothesis that the level of strategy students use to solve problems was the same across students with LD and students in Tier 2 in the third grade, fourth grade, and fifth grade. The results of the test revealed no statistically significant differences in mean ranks based on LD status in third grade ($z = -0.970$, $p > .05$), fourth grade ($z = -1.485$, $p > .05$), or fifth grade ($z = -0.456$, $p > .05$). Third-grade students with LD had an average rank of 16.88, while third-grade students in Tier 2 had an average rank of 24.13. Fourth-grade students with LD had an average rank of 7.55, while fourth-grade students in Tier 2 had an average rank of 11.07. Fifth-grade students with LD had an average rank of 7.65, while fifth-grade students in Tier 2 had an average rank of 8.70.

Operations. Qualitative analysis resulted in three levels of partitioning operations; an iterating operation was also uncovered yet considered separately: (a) partitioning with no regard to equal parts, (b) partitioning with regard to equal "halves," (c) partitioning with regard to equal parts (all cases), and (d) iteration (present/absent). Examples and descriptions follow.

Partitioning with no regard to equal parts. At this level, students partitioned objects yet did so somewhat "haphazardly," with little regard to equality of parts or exhausting the whole. When asked if it mattered that the parts were equal or if it was fair if the child produced unequal parts, the child replied, "No" or "It's OK if they are different." Ten percent of students with LD and 0% of students in Tier 2 evidenced this form of partitioning as their dominant operation.

Partitioning with regard to equal "halves." At this level, students partitioned objects and expressed specific regard to equality with respect to one-half. With partitions other than one-half, students seemed to lose equality of the parts. Twenty percent of students with LD and 0% of students in Tier 2 evidenced this form of partitioning as their dominant operation.

Partitioning with regard to equal "parts." At this level, students partitioned objects and expressed specific regard to equality with respect to all parts created. Students' attention to equality at this point is explicit and is beginning to be linked to an exhaustion of one whole. Seventy percent of children with LD and 100% of students in Tier 2 evidenced this form of partitioning as their dominant operation.

Differences between groups: Partitioning. A Mann–Whitney *U* test was conducted to evaluate the null hypothesis that the level of partitioning operations students use to solve problems is the same across students with LD and students in Tier 2 in the third grade, fourth grade, and fifth grade. The results of the test revealed no statistically significant differences in mean ranks based on LD status in third grade ($z = 0$, $p > .05$), fourth grade ($z = -1.844$, $p > .05$), or fifth grade ($z = 0$, $p > .05$). Third-grade students with LD and students in Tier 2 each had an average rank of 6.00. Fourth-grade students with LD had an average rank of 7.6, while fourth-grade students in Tier 2 had an average rank of 11.00. Fifth grade children with LD as well as students in Tier 2 had an average rank of 8.00.

Iteration. *Iteration*, as we defined it, involved a partition, yet students independently disembedded and used one piece as a stand in for all pieces. Often, students tested the piece for "correctness" against the whole in activity. Iteration was coded as present or absent holistically across the tasks. Ten percent of students with LD evidenced iteration within their problem-solving activity, while 55% of Tier 2 students used iteration.

Differences between groups: Iteration. A Mann–Whitney *U* test was conducted to evaluate the null hypothesis that the distribution of iteration operations students used to solve problems is the same across students with and without

LD in the third grade, fourth grade, and fifth grade. The results of the test revealed no statistically significant differences in mean ranks based on LD status in third grade ($z = -1.361$, $p > .05$), fourth grade ($z = -1.525$, $p > .05$), or fifth grade ($z = -1.414$, $p > .05$). Third-grade students with LD had an average rank of 5.00, while third-grade students in Tier 2 had an average rank of 6.83. Fourth-grade students with LD had an average rank of 7.85, while fourth-grade students in Tier 2 had an average rank of 10.65. Fifth-grade students with LD had an average rank of 6.00, while fifth-grade students in Tier 2 had an average rank of 9.00.

Language and representations. The analysis resulted in three levels of *language* students used to quantify the equal share: (a) *Pieces* (i.e., named share as a number of pieces), (b) *Half* (i.e., used "half" to name all unit fractions), (c) *Developing* (i.e., quantified unit fractions in activity; did not transfer this naming over to quantify amount), and (d) *Solidified* (i.e., quantified unit and nonunit fractions dependent and/or independent of activity). Three representations, (a) *Tangible*, (b) *Figurative*, and (c) *Symbolic*, were also documented. Tangible representations included concrete items, like cubes or paper. Figurative representations included drawings or the use of fingers. Symbolic representations included numeric recordings of work or verbal descriptions of symbolic representations.

Differences in groups: Language and representations. A Mann–Whitney *U* test was conducted to evaluate the null hypothesis that the distribution of language students use to solve problems is the same across students with and without LD in the third grade, fourth grade, and fifth grade. The results of the test revealed no statistically significant differences in mean ranks based on LD status in third grade ($z = -1.361$, $p > .05$), fourth grade ($z = -1.383$, $p > .05$), or fifth grade ($z = -0.834$, $p > .05$). Third-grade students with LD had an average rank of 5.80, while third-grade students in Tier 2 had an average rank of 6.17. Fourth-grade students with LD had an average rank of 7.80, while fourth-grade students in Tier 2 had an average rank of 10.71. Fifth-grade students with LD had an average rank of 7.40, while fifth-grade students in Tier 2 had an average rank of 9.20.

A Mann–Whitney *U* test was conducted to evaluate the null hypothesis that the distribution of representations students use to solve problems is the same across students with and without LD in the third grade, fourth grade, and fifth grade. The results of the test revealed no statistically significant differences in mean ranks based on LD status in third grade ($z = -0.471$, $p > .05$), fourth grade ($z = -1.102$, $p > .05$), or fifth grade ($z = -1.080$, $p > .05$). Third-grade students with LD had an average rank of 5.60, while third-grade students in Tier 2 had an average rank of 6.33. Fourth-grade students with LD had an average rank of 7.85, while fourth-grade students in Tier 2 had an average rank of 10.65. Fifth-grade students with LD had an average rank of 7.25, while fifth-grade students in Tier 2 had an average rank of 9.50.

Emergent Analysis

Key understandings. The next paragraphs describe key understandings (i.e., divisible whole, partitioning plan, notions of equality within the whole) that emerged as a result of analysis in terms of level. Then, we present the framework and the numbers of students falling at each level.

Divisible whole. Emergent analysis resulted in three levels of observed activity: (a) *No Fractions (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *Solidified (coded as 1). No Fractions* indicated students' propensity to only deal in terms of whole units—to these students, the whole is not yet conceived of as divisible, so fractions are not created. *Developing* notions of a divisible whole were viewed as a reluctant, after-the-fact notion of the whole as divisible. Students did not want to create fractional shares initially, but seemed to do so "begrudgingly." A *Solidified* notion of a divisible whole was evidenced when students readily cut apart a whole or wholes without hesitation.

Partitioning plan. Emergent analysis resulted in three levels of observed activity: (a) *No Fractions (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *A Priori Link to Sharers (coded as 1)*. *No Fractions* codes indicated students did not act on the whole in terms of partitioning. *Developing codes* meant that a plan for creating a seemingly known number of total pieces across the whole(s) is not yet anticipated or carried out in activity. Yet, students used whole numbers as a rudimentary, activity-based plan for creating fractional units within one whole (i.e., the Fry tasks) or the "leftover" (i.e., the story problems). *A priori Link to Sharers* codes indicated that students used the number of sharers to create a known number of parts. A use of multiplication, and at times division, seemed to accompany students' thinking.

Relation between partitions and parts created. Emergent analysis uncovered students' notions of a relation between partitions and parts created as *Absent* (coded as 0) or *Present (coded as 1)*. Researchers coded *Absent* if students (a) seemed to confuse cuts with parts created (e.g., to make fourths, the child folds a paper four times) or (b) made explicit statements in their activity that indicated a lack of association between parts and cuts (e.g., "three cuts for three parts"). *Present* was coded as the lack of any indication to the contrary; students' partitioning seemed to align to the number of parts they made in activity.

Level	Divisibility of the Whole	Partitioning Plan	Coordination of Equal Units Within the Whole
0: No Fractions	Will only share/deal out wholes. Whole not yet conceived as divisible. Does not act on the whole or create fractions.		
1: Early Sharer	Seems to reluctantly cut into pieces.	Trial and Error based in whole number in activity. • May begin to use "half" in activity, but it is not meaningful to the child as a quantity. It is a rudimentary sharing strategy based on two people breaking apart an item or items.	Child's attention is on making a number of parts. • Parts created are not equal in size, and the child is not bothered.
2: Half Based Sharer	Readily divides whole without hesitation.	Plan becomes evident in how students attend to sharing leftover parts or wholes after dealing out wholes or halves. • "Half" represents a meaningful quantity that the child uses to create pieces.	Begins to coordinate equal parts with exhausting the whole after they see the equal part(s) he/she created do not exhaust the whole.
3: Emerging Relational Sharer		Plans to create number of parts equal to number of sharers prior to activity. • May use knowledge of multiplication/division.	Creates equal parts while exhausting the whole by using iteration to test the part against the whole.

Figure 1. Framework of key understandings.

Exhausting the whole. Emergent analysis resulted in three levels of observed activity in terms of exhausting the whole: (a) *Early (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *Solidified (coded as 1). Early* codes indicated students' attention was solely on making equal parts. *Developing* codes indicated the child's attention shifted to the whole and was beginning to attend to sizes of equal parts within the whole, but students had difficulty making equal parts and exhausting the whole concomitantly. *Solidified codes* indicated that the child, *in activity*, coordinated making equal parts within the whole.

Notions of equality of the parts. Emergent analysis resulted in three levels of problem solving: (a) *Early (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *Solidified (coded as 1). Early* was defined as created parts unequal in size; students were not bothered by their inequality (i.e., when asked, students said that it is OK that the sizes are different or said it was "fair" to give more to one person because they are bigger). *Developing* was defined as students stating or explaining in their activity that the parts should be equal. Students paid close attention to making equal parts, yet have difficulty because they started to pay attention to the parts with respect to the whole. However, students had yet to coordinate the parts and the exhaustion of the whole. *Solidified* was defined as students paying close attention to equality of parts with respect to the whole.

A framework of key understandings. The key understandings evidenced a framework in terms of all students' initial fractional knowledge (see Note 1; see Figure 1).

Forty percent of students with LD were coded as a Level 1; 20% of students in Tier 2 were coded as Level 1. Fifty percent of students with LD were coded as a Level 2; 45% of students in Tier 2 were coded as Level 2. Ten percent of students with LD were coded as a Level 3; 35% of students in Tier 2 were coded as a Level 3.

Trend Analysis

In the final section of analysis, we present a mapping of the key understandings as evidenced by each student holistically to illuminate indicators that lead development at each level and key understandings that seemed to emerge together.

Heat map. Figure 2 illustrates a visualization of the key understandings for 39 of the 43 students (see Note 2) across the interview tasks against their overall level code. The following paragraphs describe which understandings led development at each level holistically for all interviews.

As illustrated in the heat map, seven out of 15 students holistically coded as Level 1 fully conceived of the whole as divisible (42%), while the remaining eight children's conception of a divisible whole was developing. An understanding of the need for equality of the parts was coded as developing in all but one of the students. Eleven students in Level 1 showed an early understanding of exhausting the whole when sharing. A plan for partitioning going into modeled activity was coded as early or developing in all students. Thirteen students coded at Level 1 seemed to confuse partition lines with parts in their activity. Independent iteration was not used. From the evident trends, it appears that what leads development at Level 1 include students' developing (a) notion that the whole is divisible and (b) recognition of the need for equal parts/shares; these indicators showed as "1" or "0.5" in most students.

All 18 students holistically coded at Level 2 conceived of the whole as divisible. A notion that parts need to be

Child #	Divisible Whole	Partitioning Plan	Exhaust Whole	Equality of Parts	Part/Cut Relation	Iterating	Level
1	0.5	0	0	0.5	0	0	1
4	0.5	0	0	0.5	0	0	1
5	0.5	0	0	0.5	0	0	1
6	0.5	0.5	0	0.5	0	0	1
7	0.5	0	0	0.5	0	0	1
8	0.5	0	0	0.5	0	0	1
10	0.5	0.5	0.5	0.5	0	0	1
15	1	0	0	0.5	0	0	1
21	0.5	0	0	0.5	0	0	1
22	1	0.5	0	0.5	0	0	1
23	1	0.5	0	0.5	0	0	1
24	1	0.5	0.5	0.5	0	0	1
25	1	0	0	0.5	1	0	1
29	1	0.5	0.5	1	1	0	1
30	1	0.5	0.5	0.5	0	0	1
2	1	0.5	0.5	1	1	0	2
3	1	0.5	0.5	1	1	0	2
9	1	0.5	1	1	1	0	2
11	1	0.5	0.5	1	1	0	2
12	1	0.5	0.5	1	1	0	2
13	1	0.5	0.5	1	1	0	2
18	1	0.5	0.5	1	1	1	2
19	1	0.5	0.5	1	1	0	2
20	1	1	1	1	1	0	2
32	1	0.5	0.5	1	1	1	2
33	1	0.5	0.5	1	1	0	2
34	1	0.5	0.5	1	1	1	2
35	1	0.5	1	1	1	1	2
37	1	0.5	1	1	1	1	2
38	1	0.5	0.5	1	1	1	2
40	1	0.5	0.5	1	1	1	2
41	1	0.5	1	1	1	1	2
16	1	1	1	1	1	0	2
14	1	1	1	1	1	1	3
17	1	1	1	1	1	1	3
26	1	1	1	1	1	1	3
27	1	1	1	1	1	1	3
28	1	1	1	1	1	1	3
31	1	1	1	1	1	1	3
36	1	1	1	1	1	1	3
39	1	1	1	1	1	1	3

Figure 2. Heat map.

equal becomes solid at this level of development, with all of the children evidencing recognition of the need for equal parts. The need to make equal parts, however, did not always reconcile with exhausting the whole, as students' propensity to exhaust the whole in their activity was coded as solidified in only six students. A partitioning plan ahead of modeled activity is coded as developing in all but one student. From the evident trends, it appears that what leads development at this level is not only students' evolving (a) coordination of created equal parts with exhausting the whole but also (b) an a priori plan for creating the parts.

For the eight students coded as Level 3, most key understandings were coded as solidified *in activity*. The whole is divisible for 100% of the students; parts are related to cuts for 100% of the students. A partitioning plan and exhaustion of the whole were also solidified. Moreover, Iteration occurred independently. From the evident trends, what seems to lead development in Level 3 is the testing of implicit or explicitly equal parts against the whole, seemingly in an effort to quantify the share in terms of the whole and/or to rectify creation of fractions in activity.

Discussion

Many researchers define and document initial notions of fractions through equal sharing (e.g., Empson et al., 2006;

Steffe & Olive, 2010; Streefland, 1993; Tzur, 1999, 2007). Accordingly, we utilized this literature base to design equal sharing situations to uncover initial conceptions of fractions for students with LD and mathematics difficulties. Results of the current study revealed a variety of problem-solving strategies, language, operations, and representations representative of students' initial notions of key fractional understandings. The majority of students with LD and students in Tier 2 settings evidenced partitioning in their activity. Yet, they used a rudimentary trail and error-based partitioning plan or informal notions of "halving" to partition in the equal sharing situations. Anticipations of a plan for partitioning linking the number of sharers to the whole(s) prior to activity were absent in a majority of all students in the study, LD or otherwise. Moreover, both students with LD and students in Tier 2 evidenced early notions of coordinating parts with respect to the whole, with many students attending to either the parts they created or the whole to be shared, but not both at once. Some students seemed to begin to realize the necessity of this coordination in their activity, yet few used an iterating operation to verify the coordination of the parts to the whole. Thus, our findings showed a predominance of *all* students in the study evidenced early conceptions of fractions as quantities, both within the confines of our framework and when compared with existing research that documents previous findings of students' conceptions (Empson et al., 2006; Steffe & Olive, 2010; Tzur, 2007).

In terms of differences in initial conceptions that might set students with LD apart from those of students who struggle but do not have disabilities, no significant differences based were found on any variable tested. Put differently, the level of the initial understandings of fractions, as defined in this study, evidenced by students with LD were similar to those evidenced by students who struggle to learning mathematics yet do not have disabilities. Our findings suggest that the "early" conceptions of fractional quantity uncovered in the analysis are not necessarily limited to or indicative of LD. Yet, due to the manner in which the data were analyzed (e.g., strategies, language, representations, and operations were coded for the group as a whole as opposed to students with LD and Tier 2 students separately), more research is needed to further substantiate this claim. Caution should be used in extending the findings to all students with LD.

Diminished understandings of fractions as quantities can affect students' ability to operate with fractions in higher level mathematical contexts (NMAP, 2008). In the current study, many of the students with LD and those who struggled with mathematics were able to partition, seemed to be developing plans for partitioning in their problem-solving activity, and evidenced a nascent understanding of the magnitude of parts coordinated with respect to a whole. Yet, the operational aspects involved with more advanced notions of fractions (e.g., disembedding and iterating a part to confirm it as 1/n) seemed absent or at a rudimentary level of development in many of the students we interviewed. Thus, continuing to document how this knowledge might be extended such that students may develop their conceptions of fractional quantities seems critical (Vukovic, 2012). Frameworks such as that documented in the current study may serve as a useful tool for practitioners and researchers wishing to gauge students' initial knowledge.

Limitations and Future Research

The current study has important limitations to be acknowledged. First, this study and its results need to be confirmed with additional and possibly larger samples. In our study, 43 students with LD and mathematics difficulty evidenced similar levels of reasoning while immersed in equal sharing tasks. Yet, more studies of other students with LD and mathematics difficulties' problem-solving performance is needed to further examine the claims made herein.

Another limitation rests in how the data were collected. We examined the nature of students with LD's conceptions of fractions through clinical interviews because this methodology offers flexibility and prevents researchers from simply ratifying predetermined mathematical knowledge of the researcher into an empirical study. We examined the nature of students' conceptions of fractions as evidenced through students' activity in equal sharing problems. Thus, we have no information on students' interpretations of fractions in the context of other kinds of tasks or in other data collection methods (e.g., error analysis of written assessments, etc.). Also, we did not explore any possible interaction of cognitive factors on students' problem-solving activity in the current study. It may be these factors could affect the activity of students with LD in varied ways.

Finally, due to the manner in which the data were analyzed (e.g., strategies, language, representations, and operations were coded for the group as a whole as opposed to students with LD and Tier 2 students separately), more research is needed to further substantiate claims of similarity in terms of fractions conceptions for students with LD and mathematics difficulties. Future work might work to provide a more fine-grained analysis of students' partitioning, disembedding, and iterating operations and propensity to use parts within whole conceptions of fractions along with platform tasks and situations that might work to support students with LD and mathematics difficulties to advance from parts within whole conceptions.

Implications for Practice

It is important to note that few children, LD or otherwise, develop rich conceptions of fractions as quantities in the absence of instruction that promotes its construction. For instance, Steffe (2007) estimated many students who complete the fifth grade do not have a command of the operational or mathematical underpinnings of conceiving of fractional quantities—as much as 30%, a figure that includes a majority of children without LD. Thus, it is important to focus research and practice on the development of instructional experiences that support the advancement of students' conceptions.

What might be the foundation of such instruction? To begin, we assert that instructional interventions must move away from explicit, part-of-whole based approaches found in much of the curriculum used in schools (e.g., shade parts of circular or linear wholes, write fraction names for the parts, and follow demonstrated procedures to find equivalent fractions) and move toward experiences that would build conceptions that were found to be underdeveloped in the study. Most students we interviewed in the current study were coded as a Level 1 or a Level 2 in our framework, which means that their difficulties conceptualizing fractions centered on coordinating parts with respect to a referent whole and using notions of multiplicative structures or numerical composites as templates for partitioning fractions (Hackenberg, 2013; Olive & Vomvordi, 2006). Prior research suggests many students with LD have underdeveloped multiplicative concepts and have benefited from instruction that immersed them in student-centered experiences that nurtured multiplicative reasoning (Tzur et al., 2012). To that end, planning experiences with multiplicative unit coordination in fractional situations (Tzur, 2007) could support students with LD and mathematics difficulties as they work to build their fractional knowledge.

Most students involved in the current study evidenced varying propensities to partition tangible or figurative representations of equal sharing and quantified the result as a number of pieces, halves, or fractional names for the unit and nonunit fractions they created. Lewis (2010, 2014) identified atypical ways in which two adults with LD understood fractional representations that led the participants away from understanding fractions as quantities. For example, the adults thought of shaded areas as "taken away" instead of as a part of a whole, and of one-half *not as a result* of, say, marking the whole into two equal parts but rather as the *action of marking* in and of itself. Lewis (2014) argued that their understandings, seemingly resistant to instructional intervention, reflected an "incompatibility between the student's cognitive processing and the mediated tools intended to support an understanding of fractional quantity" (p. 380).

In contrast, we did not see any evidence to suggest that students with LD conceived of partitioning the mathematical representations they utilized to solve equal sharing problems in atypical ways. Rather, due to the absence of a disembedding operation, it is more likely that many of these children hold a *parts within whole* idea of fractions as opposed to a *parts to whole* idea (Hackenberg, 2013); this may be linked to what Lewis (2014) called an atypical use of representation. Yet, from the standpoint of some researchers, a parts within whole conception is likely the result of underdeveloped yet malleable operational schemes (Hackenberg, 2013; Olive & Vomvordi, 2006) and not disability. Arguably, the conception is linked to instructional experiences that focus on shading and vocabulary part-to-whole interpretations for fractions (Olive & Vomvordi, 2006) and can indeed be difficult to overcome. Alternate starting points for fractions concepts, such as those that begin with a ratio or relation notion of iterated fractional units that are related to yet already disembedded from continuous (Frudenthal, 1983) or discrete (Olive & Vomvordi, 2006) fractional wholes, or possibly building up students' disembedding operations in whole number (Tzur & Lambert, 2011), may warrant further examination.

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Notes

1. We combined equality of parts and exhausting the whole into one category as their level descriptions were extremely connected. Moreover, Piaget, Inhelder, and Szeminska (1960) discussed equality of parts and exhausting the whole as complementary understandings.
2. Elements of data were missing for one of the five key understandings for four of the students, so they were excluded from the heat map trend analysis.

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